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Crossover from classical to quantum behavior of the Duffing oscillator through a pseudo-Lyapunov-exponent

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We discuss the quantum-classical correspondence (QCC) in a specific dissipative chaotic system, the Duffing oscillator. The quantum version of the Duffing oscillator is treated as an open quantum system and analyzed numerically by the use of quantum state diffusion (QSD). We consider a pseudo-Lyapunov exponent and investigate it in detail, varying the Planck constant effectively. We show that there exists a critical stage in which the crossover from classical to quantum behavior occurs. Furthermore, we find that a dissipation effect suppresses the occurrence of chaos in the quantum region, while it, combined with the periodic external force, plays a crucial role in the chaotic behavior of the classical system.

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Quantum-classical correspondence (QCC) is a fundamental problem in quantum mechanics. In particular, when the classical system is a chaotic one, this correspondence is still unclear; quantum dynamics (the Schrödinger equation) and the mathematical structure of quantum states (a separable Hilbert space) generally exclude the probability of chaotic phenomena [1]. Nevertheless, various studies have been done in Hamiltonian systems and are very fruitful [2]. However, chaotic behavior can occur in dissipative systems for which a definite Hamiltonian does not exist [3]. Furthermore, dissipative quantum chaos is also related to the foundation of quantum mechanics [4].

In this Rapid Communication, we discuss the QCC for a quantum version of the Duffing oscillator. The classical equation of the Duffing oscillator is $m\ddot{x}+2\gamma m\dot{x}+m\omega_0^2x^3/l^2-m\omega_0^2x=m\omega_0^2lg\cos(\omega t)$, where the scale parameter l characterizes a size of the system. If we choose a set of dimensionless parameters $(\Gamma,g,\Omega)\equiv(\gamma/\omega_0,g,\omega/\omega_0)=(0.125,0.3,1.00)$, we find chaotic motion in the Poincaré surface [5]. We treat the Duffing oscillator as an open quantum system and assume that its time evolution is described by Markovian dynamics. We assume that the system's reduced density matrix ρ evolves according to the Lindblad master equation [6],

$$\dot{\rho}=-\frac{i}{\hbar}[\hat{H},\rho]+\hat{L}\rho\hat{L}^{\dagger}-\frac{1}{2}\hat{L}^{\dagger}\hat{L}\rho-\frac{1}{2}\rho\hat{L}^{\dagger}\hat{L}. \tag{1}$$

We guess the Hamiltonian $\hat{H}$ ($\hat{H}^{\dagger}=\hat{H}$) and $\hat{L}$ in Eq. (1) phenomenologically, and analyze this system numerically by the use of quantum state diffusion (QSD) [7], a stochastic Schrödinger equation. QSD is a very effective method of numerical simulation for a system with many degrees of freedom or for a nonlinear dynamical system, since it is necessary for solving these problems to prepare many bases for representing the density matrix. We use the algorithm of the QSD given in Ref. [8]. Analyses using QSD of other dissipative systems are given in Refs. [9,10].

The quantum versions of the Duffing oscillator and similar systems have been studied by many authors. Their results support the emergence of chaotic behavior in the classical limit. Brun *et al.* [11] showed that a strange attractor occurs on the Poincaré surface. Bhattacharya *et al.* [12] showed that their result of a Lyapunov exponent is a positive value. These results are very interesting and important. However, such results are inadequate for discussing QCC because it is important to investigate what phenomena occur between the classical region and the quantum region. Therefore, we propose a method with which to discuss QCC in more detail. Let us introduce a scaling parameter β. This parameter is square of the ratio of Planck constant $\hbar$ to the value of characteristic action $S_0=ml^2\omega_0$ of the present system: $\beta^2=\hbar/S_0$. Similar parameters are introduced in Refs. [9,11]. We define the region of $\beta\sim0$ (not equal to zero) as classical and the region of $\beta=1.00$ as quantum. We investigate the system as β goes from 0 to 1 with fixed S_0. To investigate the difference in temporal behavior for two different initial conditions we calculate the following quantity:

$$\Delta(\tau)=\frac{1}{N}\sum_{\{1,2\}}\{\delta\overline{Q}_{12}(\tau)^2+\delta\overline{P}_{12}(\tau)^2\}^{1/2}, \tag{2}$$

where $\delta\overline{Q}_{12}(\tau)=\mathrm{Tr}\{\hat{Q}\rho_1(\tau)\}-\mathrm{Tr}\{\hat{Q}\rho_2(\tau)\}$ and $\delta\overline{P}_{12}(\tau)=\mathrm{Tr}\{\hat{P}\rho_1(\tau)\}-\mathrm{Tr}\{\hat{P}\rho_2(\tau)\}$. We define the dimensionless position operator, momentum operator, and time as $\hat{Q}=\hat{x}/l$, $\hat{P}=\hat{p}/ml\omega_0$, and $\tau=\omega_0t$, respectively. The canonical commutation relation for the position operator $\hat{x}$ and the momentum operator $\hat{p}$ is $[\hat{x},\hat{p}]=i\hbar$. Two density matrices $\rho_1(\tau)$ and $\rho_2(\tau)$ evolve from different initial states, $\rho_1(0)$ and $\rho_2(0)$, respectively. Hereafter, we assume that $\rho_i(0)$ $(i=1,2)$ is a pure coherent state $|\alpha_i\rangle\langle\alpha_i|$, where $\alpha_i=\sqrt{2}[\mathrm{Tr}\{\hat{Q}\rho_i(0)\}+i\,\mathrm{Tr}\{\hat{P}\rho_i(0)\}]$. The summation in Eq. (2) is over the sets of

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the chosen initial conditions, and N is the number of those sets. We calculate $\Delta(\tau)$, varying β while keeping S_0 fixed. The calculation of $\Delta(\tau)$ is similar to the derivation of the Lyapunov exponent in the classical mechanics. We call the resultant value gained from the simulation as the pseudo-Lyapunov-exponent. We show the existence of a clear crossover from classical to quantum behavior as $\beta \to 1$. Moreover, we find that the effect of dissipation in the system suppresses the emergence of chaos.

Let us explain the model for the quantum version of the Duffing oscillator. The Hamiltonian $\hat{H}$ and $\hat{L}$ in Eq. (1) are determined in order that the equations of expectation values for $\hat{Q}$ and $\hat{P}$ are equivalent to the classical equation of motion, neglecting the moments higher than second order,

$$\hat{H} = \hat{H}_D + \hat{H}_R + \hat{H}_{ex}, \tag{3}$$

$$\hat{L} = \sqrt{\Gamma}(\hat{Q} + i\hat{P}), \tag{4}$$

where $\hat{H}_D = \hat{P}^2/2 + \beta^2\hat{Q}^4/4 - \hat{Q}^2/2$, $\hat{H}_R = \Gamma(\hat{Q}\hat{P} + \hat{P}\hat{Q})/2$, $\hat{H}_{ex} = -g\hat{Q}\cos(\Omega t)/\beta$, and $[\hat{Q}, \hat{P}] = i$. The dimensionless parameters Γ, g, and Ω are the same as those of the classical equation of motion $[(\Gamma, g, \Omega) = (0.125, 0.3, 1.00)]$. Notice that β is introduced naturally when the operators are transformed to dimensionless expressions. The first term and the remainder in $\hat{H}_D$ represent the kinetic term and the double-well potential term, respectively. The second term in Eq. (3), $\hat{H}_R$, represents the strength renormalization for the coupling of interaction between system and environment. We can remove this term from Eq. (3) by making the transformation $\hat{Q} \to \hat{Q}$ and $\hat{P} \to \hat{P} - \Gamma\hat{Q}$. The third term in Eq. (3), $\hat{H}_{ex}$, represents the potential term corresponding to the periodic external force. Generally, the generator of Eq. (1) is independent of the time. We determine it simply so as to reproduce the external force in the equation of expectation values. The definition of β implies that the value of l, which characterizes a size of the system, for β_1 is larger than one for β_2 if $\beta_1 < \beta_2$. The value of the potential around the origin becomes larger as β goes to zero. Before numerical simulations can begin, we have to determine a suitable value of $\epsilon \equiv \Delta(\tau = 0)$. Notice that two points in the phase space are not *distinguishable* from the view of quantum mechanics, if they coexist inside the same Planck cell. The size of the Planck cell is determined by the Heisenberg's uncertainty relation. In this model, the commutation relation $[\hat{Q}, \hat{P}] = [\hat{x}, \hat{p}]/S_0 = i\beta^2$ is fulfilled. Then, the Planck cell has a constant volume of $\Delta Q \Delta P = \beta^2/2$ in the scaled phase space, whereas it has $\Delta x \Delta p = \hbar/2 = \beta^2 S_0/2$ in the original phase space. With the fixed value of typical action S_0 for the system, the smaller β^2 corresponds to the smaller $\hbar$; the system exhibits more classical behavior. We define an effective Planck cell as a region in the $(\mathrm{Tr}(\hat{Q}\rho), \mathrm{Tr}(\hat{P}\rho))$ plane whose length is β in units of $\sqrt{S_0/2}$. Then, we investigate two choices for ϵ: (1) $\epsilon = 0.01$ (fixed), where two points in the phase space are *distinguishable* only for the classical region ($\beta = 0.01$); (2) $\epsilon \sim \beta$, where the points are *distinguishable* for all βs. As a first step of our simulation, we produced results similar to those of Ref. [11] for the constant phase maps and verified the existence of the strange attractor. We have randomly chosen initial states, using the data calculated for constant phase maps. Each initial coherent state chosen gives an initial value of the action. However, the present case is a dissipative system under a periodic external force. Therefore, the initial condition does not affect our results.

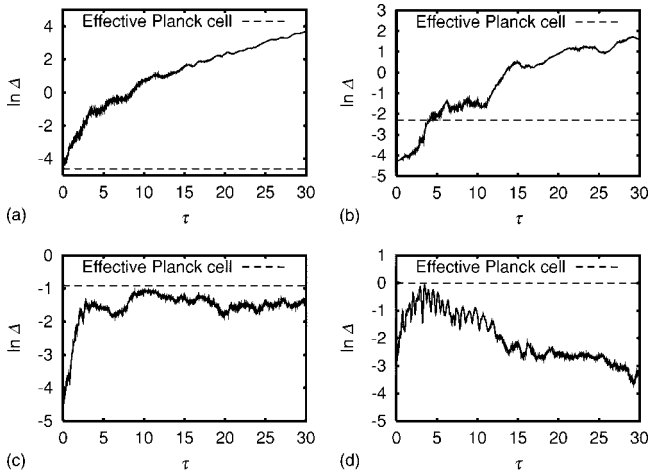

FIG. 1. The time evolution of $\Delta(\tau)$ with ϵ fixed as 0.01. The quantities plotted are dimensionless by definition. The complex Wiener process is used in the QSD. (a) and (b) are obtained with a single realization of the complex Wiener process for each initial condition (20 samples). (c)–(d) are obtained by averaging over 100 realizations of the complex Wiener process for each initial condition (10 samples). (a)–(d) are for $\beta = 0.01$, 0.10, 0.40, and 1.00, respectively.

We show the results of simulating $\Delta(\tau)$ with ϵ fixed as 0.01. In Fig. 1(a), we find an exponential increase of $\Delta(\tau)$, a characteristic behavior of chaos. This corresponds to the fact that the maximal Lyapunov exponent is positive in classical chaotic systems. This behavior is also consistent with the existence of the strange attractor in Ref. [11], and verifies that the quantum version of the Duffing oscillator maintains chaotic behavior for $\beta = 0.01$. In Fig. 1, we see very different behavior between (b) and (c)–(d). For these β values, each pair of initial points is within the same Planck cell and is *indistinguishable* from each other. However, the value of $\Delta(\tau)$ for $\beta = 0.10$ increases gradually, and crosses the size of the effective Planck cell after some duration. This suggests that a remnant of chaotic dynamics still survives for $\beta = 0.10$. We find that this behavior continues more or less up to $\beta = 0.40$ through more detailed numerical results. On the other hand, the values of $\Delta(\tau)$ for both $\beta = 0.40$ and 1.00 are always less than the size of effective Planck cell; the chaotic dynamics has been completely lost. This observation suggests that the crossover from classical to quantum behavior exists around $\beta \sim 0.40$. In the above argument, we have to note the points $(\mathrm{Tr}\{\hat{Q}\rho\}, \mathrm{Tr}\{\hat{P}\rho\})$ in the constant phase map for each β are distributed in a bounded domain, which forces the value of $\Delta(\tau)$ for every β to reach a saturation value for a long-time duration. We estimate the maximum value of the distance between two points in the bounded domain, which is written by L, by our constant phase map data: $(\beta, \ln L)$

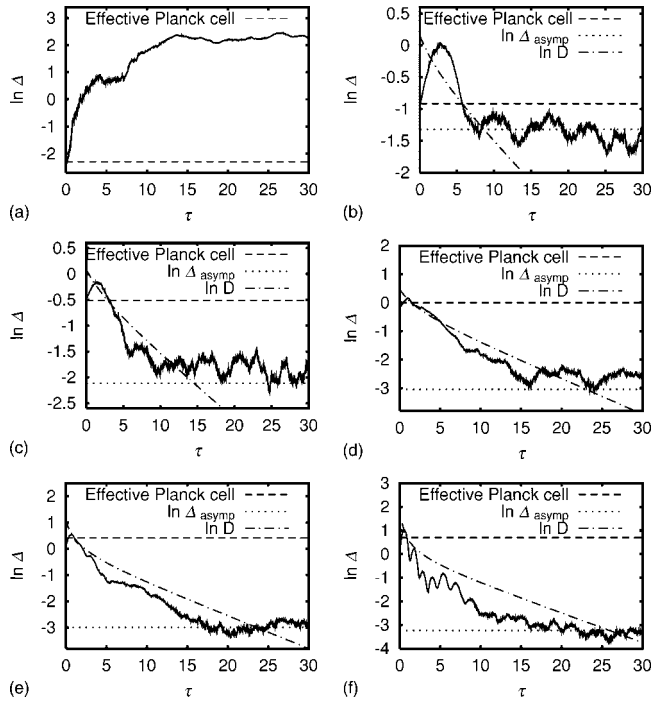

FIG. 2. The time evolution of $\Delta(\tau)$ with $\epsilon=\beta$. The quantities plotted are dimensionless by definition. The complex Wiener process is used in the QSD. The asymptotic value of $\Delta(\tau)$, Δ_{asymp}, is indicated by the dotted line. The right-hand side of Eq. (7), $D(\tau)$, is expressed by the broken dotted line. Figure (a) is obtained with a single realization of the complex Wiener process for each initial condition (20 samples). Figures (b)–(f) are obtained by averaging over 100 realizations of the complex Wiener process for each initial condition (10 samples). Figures (a)–(e), and (f) are for $\beta=0.10$, 0.40, 0.60, 1.00, 1.50, and 2.00, respectively.

$=(0.01, 5.62)$, (0.10, 3.32), (0.40, 1.90), and (1.00, 0.59). The logarithmic value of the saturation for $\beta=0.01$, for example, is estimated at about 5 through the simulation up to $\tau=60$; this is almost equal to $0.8 \ln L$. This type of saturation due to the bounded domain is different from the results in Figs. 1(c) and 1(d). Therefore, the comparison of the value of $\Delta(\tau)$ with the size of the effective Planck cell is sufficient to calculate up to $\tau=30$.

Let us show the results for $\epsilon\sim\beta$, where the initial two points are separated by the effective Planck cell size. We compute $\Delta(\tau)$ for $\beta=0.10$, 0.40, 0.60, 1.00, 1.50, and 2.00. The behavior of $\Delta(\tau)$ for $\beta=0.10$ [Fig. 2(a)] is an exponentially increasing one, which is similar to the result in Fig. 1(b). In Figs. 2(b)–2(f), we find that, except for a very short period after the starting time, the value of $\Delta(\tau)$ for each β decreases for some duration and tends to approach a certain constant value Δ_{asymp} asymptotically, which is indicated by the dashed lines. Let us write τ_0 for when the value of $\Delta(\tau)$ becomes smaller than the size of the effective Planck cell. Hereafter, we use these values to characterize the result in Figs. 2(b)–2(f). Thus, these observations again allow us to consider that there is a crossover from classical to quantum behavior around $\beta\sim 0.40$. These results suggest that the quantum version of the Duffing oscillator lies in the classical region for $\beta=0.01$ and 0.10, in the quantum region for $\beta=1.00$, and in the crossover from classical to quantum behavior around $\beta\sim 0.40$. Furthermore, let us call the case of $\beta>1.00$ the deep quantum region.

We investigate the results for $\beta\geqslant 0.40$ and $\epsilon\sim\beta$ in detail. By the use of the results in Figs. 2(b)–2(f), the values of τ_0 and Δ_{asymp} are estimated as follows: $(\beta, \tau_0, \Delta_{asymp}) = (0.40, 5.71, -3.75)$, $(0.60, 2.88, -8.30)$, $(1.00, 1.45, -21.0)$, $(1.50, 1.04, -20.0)$, and $(2.00, 0.91, -25.4)$. When a density matrix is represented by the coherent state $|\alpha\rangle$, its diagonal elements are real valued functions of the complex α plane. In this sense we regard $\Delta(\tau)$ as the distance between the centers of $\rho_1(\tau)$ and $\rho_2(\tau)$ on the complex α plane. On the other hand, the distribution of $\rho_i(\tau)$ $(i=1,2)$ itself spreads generally wider than the region given by the minimal uncertainty state. For a time longer than τ_0, the values of $\Delta(\tau)$ for $\beta>0.40$ are less than the size of the effective Planck cell, as indicated by each dotted line in Figs. 2(b)–2(f). Through such consideration, we suppose that the value of $\Delta(\tau)$ is less than the spread of $\rho_i(\tau)$; the difference between $\rho_1(\tau)$ and $\rho_2(\tau)$ is not so large. Therefore, we approximate $\Delta(\tau)$ by the quantity characterizing the spread coming from a single $\rho(\tau)$ after τ_0. We assume that this is expressed by a quantity Σ introduced in Ref. [7],

$$\Delta(\tau) \approx \int_V dqdp\mu(q,p)\Sigma(q,p,\tau), \tag{5}$$

where q, p are characteristic parameters of initial pure coherent state $[\alpha=\sqrt{2}(q+ip)]$, $\mu(q,p)$ is the distribution function of the initial conditions, and V is the region of integration; the size of V is L. The spread is given by

$$\Sigma^2 \equiv M\{\sigma(\hat{a}^\dagger,\hat{a})\} \equiv M\{\langle\hat{a}^\dagger\hat{a}\rangle - \langle\hat{a}^\dagger\rangle\langle\hat{a}\rangle\}, \tag{6}$$

where $\hat{a}=(\hat{Q}+i\hat{P})/\sqrt{2}$, the symbol $\langle\hat{O}\rangle$ is the expectation value of an operator $\hat{O}$ for a stochastic state vector used in the formulation of the QSD [7], and the symbol M represents the mean over the ensemble of stochastic processes in the QSD. Meanwhile, the observation of $\Delta(\tau)$ in Figs. 2(b)–2(f) suggests that each $\Delta(\tau)$ simply decreases until it reaches Δ_{asymp}. Percival [7] showed that the diffusion due to Lindblad operators dominates for the shorter time, whereas the drift due to $\hat{H}$ dominates for the longer time. Here, we assume simply that the effect of $\hat{H}$ is negligible until $\Delta(\tau)$ reach Δ_{asymp}. We get approximately an inequality involving $\Sigma^2(q,p,\tau)$ and $\partial\Sigma^2/\partial\tau(q,p,\tau)$ using the QSD equation. Performing the integral with respect to the dimensionless time from τ_0 to τ in both sides of it, we obtain an inequality for $\Sigma(q,p,\tau)$. Substitute it into Eq. (5) and assuming that $\mu(q,p)$ is a uniform distribution function and $\Sigma(q,p,\tau=\tau_0)=\epsilon(\sim\beta)$, the upper bound for $\Delta(\tau)$ is obtained as follows:

$$\Delta(\tau-\tau_0) \lesssim \left[\left(1+\frac{1}{\beta^2}\right)e^{2\Gamma(\tau-\tau_0)}-1\right]^{-1/2}. \tag{7}$$

Let us denote the time when $\Delta(\tau)$ has reached Δ_{asymp} as τ_{asymp}. Curves calculated by the right-hand side of Eq. (7), $D(\tau)$, are represented by broken dotted lines in Fig. 2. In this figure, we find that in the case of $\beta\geqslant 1$, Eq. (7) is a good

approximation of the upper bound for any τ between τ_0 and τ_{asymp}. This is not appropriate for $\beta=0.40$ and 0.60, since the verification of Eq. (5) is subtle in such cases. The value of $\Delta(\tau)$ keeps a constant value Δ_{asymp}, once it reaches this value. This constant value is probably related to the time evolution of system due to $\hat{H}$, but we do not know an analytical method to determine its value. In the classical mechanics, a dissipative chaotic system generates chaotic dynamics due to the coexistence of the dissipative effect and periodic external force. In the quantum version of the Duffing oscillator, we draw the following conclusion: If S_0 is much greater than $\hbar$, the system is similar to the classical one and the existence of dissipation is very important for occurrence of chaotic dynamics. On the other hand, if S_0 is smaller than $\hbar$ (i.e., in the quantum and deep quantum cases), the above analysis suggests that the effect of dissipation suppresses even the occurrence of chaotic behavior.

In the present work, we find the crossover from classical to quantum behavior in the quantum version of the Duffing oscillator through the analysis of $\Delta(\tau)$. This method is expected to be an effective one for investigating QCC in dissipative chaotic systems.

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